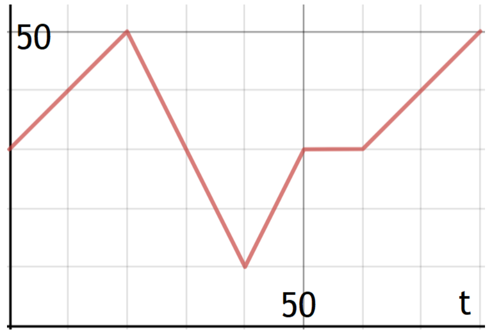
MATH 102:107, CLASS 19 (FRI OCT 20)



(1) The level of pollution in a lake is dependent on the population of humans by the lake. Let $P(H) = H^2$ equal the amount of human-created pollution, where H is the number of humans (in thousands). Regular census-taking yields the graph of $y = H(t)$ shown at right. (t in years)

We would like to understand how the pollution levels change with time.

(a) Calculate $\frac{dP}{dt}$ at $t = 30$.

Solution: $\frac{dP}{dt} = \frac{dP}{dH} \cdot \frac{dH}{dt} = 2H(30)H'(30) = 2(30)(-2) = -120$

(b) Calculate $\frac{dP}{dt}$ at $t = 10$.

Solution: $\frac{dP}{dt} = 2H(10)H'(10) = 2(40)(1) = 80$

(c) Calculate $\frac{dP}{dt}$ at $t = 55$.

Solution: $\frac{dP}{dt} = 2H(55)H'(55) = 2(30)(0) = 0$

(2) You want to study the survival rate of bacteria under exposure to the antibiotic *Penicillin*. You expose several populations of bacteria, each starting with 10^7 cells, to different concentrations of the compound and observe the results. You would like to find a statistical model which fits the data.

Concentration (x)	2	3	4	5	6
# Survivors (y)	20000	1600	160	80	32

(a) Why would it be a bad idea to fit the data with a line $y = ax$?

Solution: When the concentration is $x = 0$, we would expect the number of survivors to be $y = 10^7$, NOT $y = 0$. As x increases, the number of survivors should decrease, NOT increase. And as the concentration goes to ∞, the number of survivors should tend to 0, Not ∞. On all three fronts, the behavior of a line $y = ax$ with $a > 0$, doesn't make sense given the real-life situation we are modeling!
One type of function which DOES line up with what we expect for the real-life situation, is *exponential decay*, i.e. a function of the form $y = C \cdot 10^{-ax}$

for constants C and a. In this case, we will pick $C = 10^7$, because when $x = 0$, we want $y = 10^7$. Fitting the data above with an equation of this form, is equivalent to replacing the y-values by $-\log_{10}(y/10^7)$, which we do in the next part.

(b) You instead replace the y-values with the *negative logarithm* of the *proportion* of survivors. Your table now looks like this

Concentration (x)	2	3	4	5	6
-log(Proportion survivors) (y)	2.7	3.8	4.8	5.1	5.5

Find the slope a which gives the best-fit line $y = ax$. (You don't have to simplify your expression.)

Solution: Recall that the best-fit line minimizes the sum of the squared residuals:

$$SSR(a) = (y_1 - ax_1)^2 + \ldots + (y_n - ax_n)^2$$
$$SSR'(a) = -2x_1(y_1 - ax_1) - \ldots - 2x_n(y_n - ax_n)$$

Set this equal to zero to get

$$a = \frac{x_1y_1 + \ldots + x_ny_n}{x_1^2 + \ldots + x_n^2}$$

In our particular case, we get

$$a = \frac{2(2.7) + 3(3.8) + 4(4.8) + 5(5.1) + 6(5.5)}{2^2 + 3^2 + 4^2 + 5^2 + 6^2} = \frac{21}{20}$$

(3) (NOTE: On the in-class version of this worksheet, I made a typo. It is corrected here.) An animal is deciding what proportion of its food-gathering time, x, it should allot between two different types of food (where $0 < x < 1$).

(a) Suppose there are two types of food, 1 and 2, and the nutrition gained from spending x portion of time on each is $F_1(x) = x^{1/2}$ and $F_2(x) = Nx$ for some positive constant N. What is the maximum amount of nutrition the animal can gain, and for what value of x does this happen? Your answer will depend on N.

Solution: If the animal spends x of its time on the first type of food, and $1-x$ on the second, then its total nutritional gain is $F(x) = x^{1/2} + N(1-x)$. Find the critical points:

$$F'(x) = \frac{1}{2\sqrt{x}} - N = 0 \implies \sqrt{x} = \frac{1}{2N} \implies x = \frac{1}{4N^2}$$

This lies in the interval $[0, 1]$ if and only if $N \geq 1/2$. To determine if this is a maximum, we can either plug back into $F(x)$, or calculate the second derivative. We will do the first option.

$$F(0) = 0^{1/2} + N(1-0) = N \qquad F(1) = 1^{1/2} + N(1-1) = 1$$

$$F(1/4N^2) = \frac{1}{2N} + N\left(1 - \frac{1}{4N^2}\right) = N + \frac{1}{2N}$$

Therefore, $x = 1/4N^2$ is the maximum when $N \geq 1/2$, and otherwise, the maximum occurs at $x = 1$.

(b) Same question, but for $F_1(x) = x^2$ and $F_2(x) = Nx$.

Solution: Now, $F(x) = x^2 + N(1 - x)$. Find the critical points:

$$F'(x) = 2x - N = 0 \implies x = N/2$$

This is a critical point in the interval if and only if $N \leq 2$. We plus back into $F(x)$ to see if this is a maximum

$$F(0) = 0^2 + N(1 - 0) = N \qquad F(1) = 1^2 + N(1 - 1) = 1$$

$$F(N/2) = (N/2)^2 + N(1 - N/2) = \frac{N^2}{4} + N - \frac{N^2}{2} = N - \frac{N^2}{4}$$

Therefore, this is a local minimum! Thus, the maximum value on the interval $[0, 1]$ is always at $x = 0$.